FOUR-LEVEL SECOND ORDER ROTATABLE DESIGNS FROM PARTIALLY BALANCED ARRAYS

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1. INTRODUCTION

Four-level and six-level second order rotatable designs (SORD) are the recent explorations in the field of response surface designs. Nigam and Dey (1970) and Dey (1970) gave methods of construction of 4-level and 6-level SORD. Earlier, Mehta (1964) obtained a 4-level and a 6-level SORD for 4 factors each through operating orthogonal transformation on the SORD derived from the balanced incomplete block (BIB) desigh with parameters $v=b=4$, $r=k=3$, $\lambda=2$. Recently, Gupta (1971) obtained some series of 4-level SORD obtainable from the pair-wise balanced designs by omitting a column from the incidence matrices of BIB designs. Gupta (1971) has also given a method of constructing 6-level SORD from 3-symbol partially balanced (PB) arrays of strength two.

Dey (1970) obtained 4-level SORD using unit matrices, Nigam and Dey (1970), practically following similar lines, and Gupta (1971), using incudence matrices of BIB designs. The seemingly different approaches followed by different authors so far may, obviously, be looked upon as particular cases of a unified approach of constructing these designs through the uses of partially balanced arrays of strength two (Chakravarty, 1956). An attempt in this direction, to evolve a unified approach is, therefore, worthwhile.

In this paper, we derive certain conditions to be satisfied by 2-symbol PB arrays of strength two from which 4-level SORD can, in general, be constructed. Utilizing these results, we shall be able to construct several new series of 4-level SORD's. In fact, we are led to the interesting result that a 4-level SORD for (*i*) $v-x$ factors from b magnitude sets, and (*ii*) v factors from $b+c$ magnitude sets can always be constructed from a BIB design with parameters v, b, r, k and λ, if it satisfies the condition ; r greater than 3λ, or $5r-2b-3\lambda$ is greater than 0 ($x > 0$, $c > 0$). The latter designs are obtained by taking suitably chosen magnitude sets in addition to the sets obtainable from the incidence matrices of BIB designs.

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2. Partially Balanced (PB) Arrays of Strength Two

If there are n factors each at s levels then there are altogether s^n assemblies (design points or treatment combinations) of which a subset of N assemblies is known as an array.

An array involving n factors $F_1, F_2 \ldots\ldots F_n$ each with s levels has been defined as a partially balanced array of strength two, if for any group of 2 factors a combination of levels of 2 factors, F_{1i_1} F_{2i_2} occurs $\lambda_{i_1 i_2}$ times where $\lambda_{i_1 i_2}$ remains the same for all permutations of a given set (i_1, i_2) and for any group of 2 factors, i_j ranging from 0 to $(s-1)$ for all j (Chakravarty, 1956). This is also called an s-symbol PB array of strength two.

Let us consider a 2-symbol PB array of strength two for v factors in b assemblies, with α and β as symbols. Let, then, the pair-wise frequencies be $\lambda_{\alpha\alpha}$, $\lambda_{\alpha\beta}=\lambda_{\beta\alpha}$ and $\lambda_{\beta\beta}$. If, now, using each row of this array as a "magnitude set" a design is constructed, following Das (1961), by "multiplying each of these magnitude sets with a suitably chosen set of factorial treatment combination in two levels -1 and $+1$", then one can easily see that the design so obtained becomes a 4-level SORD in level $\pm\alpha$ and $\pm\beta$, provided the following condition (derivable from the 'rotatability' condition) is satisfied :

$$r_\alpha \alpha^4 + r_\beta \beta^4 = 3(\lambda_{\alpha\alpha}\alpha^4 + 2\lambda_{\alpha\beta}\alpha^2\beta^2 + \lambda_{\beta\beta}\beta^4), \qquad \ldots(3\cdot1)$$

where $\quad r_\alpha = \lambda_{\alpha\alpha} + \lambda_{\alpha\beta}, \;$ and $\; r_\beta = \lambda_{\beta\alpha} + \lambda_{\beta\beta}.$

Assuming $\alpha \neq 0$, $\beta \neq 0$, condition (3·1) can be shown to be equivalent to the following quadratic equation in an arbitrary variable $x = \dfrac{\beta^2}{\alpha^2}$:

$$(r_\beta - 3\lambda_{\beta\beta})x^2 - 6\lambda_{\alpha\beta}x + (r_\alpha - 3\lambda_{\alpha\alpha}) = 0. \qquad \ldots(3\cdot2)$$

Equation (3·2) yields :

$$x = \frac{\beta^2}{\alpha^2} = \frac{6\lambda_{\alpha\beta} \pm \sqrt{36\lambda^2_{\alpha\beta} - 4(r_\alpha - 3\lambda_{\alpha\alpha})(r_\beta - 3\lambda_{\beta\beta})}}{2(r_\beta - 3\lambda_{\beta\beta})} \qquad \ldots(3\cdot3)$$

Let us now examine under what conditions (3·3) will give us an (or two) admissible solutio (s) for $\dfrac{\beta^2}{\alpha^2}$. The following cases may arise:

Case I. Let $r_\beta < 3\lambda_{\beta\beta}$. Then, to have an admissible solution for $\dfrac{\beta^2}{\alpha^2}$, from (3·3), we must have

$$36\lambda^2_{\alpha\beta} - 4(r_\alpha - 3\lambda_{\alpha\alpha})(r_\beta - 3\lambda_{\beta\beta}) > 36\lambda^2_{\alpha\beta},$$

that is, $\quad (r_\alpha - 3\lambda_{\alpha\alpha}) > 0, \qquad \ldots(3\cdot4)$

Case II. Let $r_\beta=3\lambda_{\beta\beta}$. In this case, equation (3·2) reduces to :

$$6\lambda_{\alpha\beta}x-(r_\alpha-3\lambda_{\alpha\alpha})=0. \qquad ...(3\cdot5)$$

Therefore, $\left(\frac{\beta^2}{\alpha^2}\right)=\left(\frac{r_\alpha-3\lambda_{\alpha\alpha}}{6\lambda_{\alpha\beta}}\right)$

This shows that when $r_\beta=3\lambda_{\beta\beta}$, an admissible solution for $\frac{\beta^2}{\alpha^2}$ will be available only when

$$(r_\alpha-3\lambda_{\alpha\alpha})>0. \qquad ...(3\cdot6)$$

Case III. Let $r_\beta>3\lambda_{\beta\beta}$. In this situation, one can show that

$$36\lambda^2_{\alpha\beta}-4(r_\alpha-3\lambda_{\alpha\alpha})(r_\beta-3\lambda_{\beta\beta})>0$$

is always true, for, then,

$$9\lambda^2_{\alpha\beta}-(r_\alpha-3\lambda_{\alpha\alpha})(r_\beta-3\lambda_{\beta\beta})$$

$$=9\lambda^2_{\alpha\beta}-(\lambda_{\alpha\beta}-2\lambda_{\alpha\alpha})(\lambda_{\alpha\beta}-2\lambda_{\beta\beta})$$

$$=8\lambda^2_{\alpha\beta}-2\lambda_{\alpha\beta}(\lambda_{\alpha\alpha}+\lambda_{\beta\beta})-4\lambda_{\alpha\alpha}\lambda_{\beta\beta}>32\lambda^2_{\beta\beta}+4\lambda_{\beta\beta}(\lambda_{\alpha\alpha}+\lambda_{\beta\beta})-4\lambda_{\alpha\alpha}\lambda_{\beta\beta},$$

i.e., $\qquad >36\lambda^2_{\beta\beta}.$

$(\because \qquad \lambda_{\alpha\beta}>2\lambda_{\beta\beta})$

Thus, when $r_\beta>3\lambda_{\beta\beta}$, a solution for $\frac{\beta^2}{\alpha^2}$ is always available. Taking the negative value of the square-root in (3·3), an alternative solution of $\left(\frac{\beta^2}{\alpha^2}\right)$ can be obtained only when $r_\beta>3\lambda_{\beta\beta}$ and $r_\alpha>3\lambda_{\alpha\alpha}$. This is so because then we have

$$36\lambda^2_{\alpha\beta}>36\lambda^2_{\alpha\beta}-4(r_\alpha-3\lambda_{\alpha\alpha})(r_\beta-3\lambda_{\beta\beta}).$$

We thus see that if $r_\alpha>3\lambda_{\alpha\alpha}$, then a PB array will always lead to an SORD.

To sum up, we have the following:

Theorem 1. A 2-symbol partially balanced array of strength two in v factors must satisfy

either (*i*) $\lambda_{\alpha\beta}>2\lambda_{\alpha\alpha}$

or (*ii*) $\lambda_{\alpha\beta}>2\lambda_{\beta\beta}$

to yield a four-level second order rotatable design through the method of construction described in this section. In case the design points of the SORD *happen to be 'equidistant'*, we omit x columns suitably from it, following Nigam and Dey (1970), to obtain a 4-level SORD for v-x factors ; or alternatively, we may omit x columns out of the v columns of the PB array itself, and then "multiply" the magnitude sets obtained from it with factorial combinations to get 4-level SORD for v-x factors.

Proof. From a consideration of the three cases :

(*i*) $r_\beta > 3\lambda_{\beta\beta}$, (*ii*) $r_\beta = 3\lambda_{\beta\beta}$ and (*iii*) $r_\beta < 3\lambda_{\beta\beta}$ described earlier, the theorem becomes evident because $r_\beta = \lambda_{\beta\beta} + \lambda_{\beta\alpha}$, and $r_\alpha = \lambda_{\alpha\alpha} + \lambda_{\alpha\beta}$.

Example 1. Consider the PB arrays used by Dey (1970) to construct 4-level SORD. In such cases,

$$\lambda_{\alpha\alpha}=0,\ \lambda_{\alpha\beta}=\lambda_{\beta\alpha}=1,\ \lambda_{\beta\beta}=v-1.$$

Obviously, this satisfies $\lambda_{\alpha\beta} > 2\lambda_{\alpha\alpha}$, and hence these PB arrays will always lead to 4-level SORD's.

4. 4-LEVEL SORD FROM BIB DESIGNS

It is known that the incidence matrix of any balanced incomplete block (BIB) design is a 2-symbol (symbols are 0 and 1) PB array of strength two. Let us replace 1's and 0's in this incidence matrix by α's and β's respectively. We can now use this PB array with α, β as symbols, to construct a 4-level SORD by applying theorem 1. We are thus led to the following interesting corollaries.

Corollary 1. A BIBD with parameters v, b, r, k, λ must satisfy either (*i*) $r > 3\lambda$, or (*ii*) $5r-3b-3\lambda > 0$, to yield a 4-level SORD through the present method of construction.

This is so because, here, we have :

$$\lambda_{\alpha\alpha}=\lambda,\ \lambda_{\alpha\beta}=\lambda_{\beta\alpha}=r-\lambda,\ \lambda_{\beta\beta}=b\text{-}2r+\lambda.$$

Corollary 2. The series of symmetrical BIB designs having the parameters $v=b=4\lambda+3$, $r=k=2\lambda+1$, λ, where v is an odd prime or prime power, do not yield 4-level SORD through the present method of construction.

Corollary 3. Through the method of construction in consideration, no BIB design having $r=3\lambda$ will lead to any 4-level SORD. This is due to the fact that for every BIBD, $r=3\lambda$ implies that $5r-2b-3\lambda \leqslant 1$, because of the inequality

$$b \geqslant 3(r-\lambda)$$

proved in Dey and Das (1970).

Since the resulting design points from a BIBD will always be 'equidistant', we have to omit one or more columns (corresponding to factors) from the array of design points or the PB array itself as suggested in theorem 1.

An alternative procedure which will give SORDs for v, and not $v—x$, factors, would be to augment the PB array by some more assemblies so that the occurrence of 'equidistance' is undone and at the same time it remains a PB array satisfying the necessary conditions, and then multiply the corresponding magnitude sets by factorial treatment combinations. Suppose, to a given PB array with frequency parameters $\lambda_{\alpha\alpha}$, $\lambda_{\alpha\beta}$ and $\lambda_{\beta\beta}$, we add n_α assemblies of the type $(\alpha\alpha\alpha...\alpha)$ and n_β assemblies of the type $(\beta\beta\beta...\beta)$. The new PB array will, then, have to satisfy

either (*i*) $\lambda_{\alpha\beta}>2\lambda_{\alpha\alpha}+n_\alpha$,

or (*ii*) $\lambda_{\alpha\beta}>2\lambda_{\beta\beta}+n_\beta$.

to yield a 4-level SORD. If the original PB array is taken from a BIBD having the parameters v, b, r, k, λ, then, by adding n_α $(\alpha\alpha...\alpha)$ type and n_β $(\beta\beta...\beta)$-type magnitude sets, (n_α and n_β are not simultaneously equal to zero), we can construct a 4-level SORD, provided we have :

either (*i*) $r—3\lambda>n_\alpha$,

or (*ii*) $5r—?b—3\lambda>n_\beta$.

This leads us to the following corollaries which help us to construct 4-level SORD's in smaller number points through the method of augmentation of magnitude sets in a PB array, as described above.

Corollary 4. If a BIBD having the parameters v, b, r, k, λ satisfies $r>3\lambda$, then choosing $n_\alpha=0$ and $n_\beta=1$, we can construct a 4-level SORD through the method of augmentation of magnitude sets.

Corollary 5. If a BIBD having the parameters v, b, r, k, λ satisfies $5r-2b—3\lambda>0$, then choosing $n_\alpha=1$, $n_\beta=0$, we can construct a 4-level SORD through the method of augmentation of magnitude sets.

Making use of the results established thus far, the following new series of 4-level SORD for v factors from $v+1$ magnitude sets, is obtained.

A SERIES OF 4-LEVEL SORD

It is easily seen that the symmetrical series of BIB designs with the parameters $v=b=s^2+s+1$, $r=k=s+1$, $\lambda=1$, with $n_\alpha=0$, $n_\beta=1$ yield, by virtue of corollary 4, a series of 4-level SORD for all $s>2$ (s is a prime or a prime power).

5. Remarks

Alternative designs having same number of points as those of Dey (1970) are obtained through the present study. Also, the results of Gupta (1971) on construction of 4-level SORD's from BIB designs are proved pretty easily in this paper. In fact, they are particular cases of the general results established herein.

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Summary

Conditions to be satisfied by two-symbol partially balanced arrays of strength two, to yield four-level second order rotatable designs (SORD), are derived. Application is then made to construct SORD, from BIB designs through suitable augmentation of certain magnitude sets. A new series of four-level SORD for v factors from $v+1$ magnitude sets, when v is of the form s^2+s+1 (s being a prime or a prime power), is reported.

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